M121 3.4 Further Applications of Optimization

Objectives

1) Solve optimization problems

Key steps:

1) Identify the objective and the question, and any constraints.
2) Write down the concepts and how they are constructed.
3) Understand the dynamic
4) Define a variable
5) Express concepts using defined variable
6) Simplify objective function. (*)
7) Find relative extrema using 1st or 2nd derivative test.
8) Answer the question, with units

(*) If necessary, find a constraint equation
solve it for a variable, and
substitute result into objective function.

① An orange grower observes that planting 80 trees per acre yields 60 bushels of oranges per tree, and estimates that for each additional tree per acre, yield is decreased 2 bushels per tree.
How many trees should be planted per acre to maximize harvest?

Objective: Harvest (max)
Question: # trees per acre.

Concepts: Harvest $= (\text{yield per tree}) \cdot (\#\ \text{trees}/\text{acre})$

Understand the dynamic
if plant 1 more tree/acre. $\longleftrightarrow$ reduce yield 2 bushels/tree

If increase once $(80+1)$ trees/acre $\longleftrightarrow$ $(60-2)$ bushels/tree
twice $(80+2)$ $\longleftrightarrow$ $(60-2\cdot 2)$
thrice $(80+3)$ $\longleftrightarrow$ $(60-3\cdot 2)$

If $x =$ # times we increase by 1 tree
trees per acre $= 80+x$
yield per tree $= 60-2x$

Harvest $=$ (yield per tree)(# trees/acre)

$H(x) = (60-2x)(80+x)$

$= 4800 + 60x - 160x - 2x^2$ FOIL

$H(x) = 4800 - 100x - 2x^2$ combine

$H'(x) = -100 - 4x$

$H'(x) = 0$

$-100 - 4x = 0$

$-100 = 4x$

$-25 = \frac{-100}{4} = x$

$H''(x) = -4$ $\frown$ max

trees per acre $= 80 + x$

$= 80 + (-25)$

$= \boxed{55 \text{ trees/acre}}$

(2) An automobile dealer can sell 4 cars/day at a price of \$12,000/car. They estimate that for each \$200 reduction in price, they can sell 2 more cars per day. If each car costs \$10,000 and fixed costs are \$1000, what price should they charge to maximize profit?

objective: profit (maximum)

question: price per car.

> Concepts: Profit = Revenue − Costs
>
> Revenue = (# sold)·(price per item)
>
> Costs = Fixed costs + (# sold)(cost per item)

Substitute given values:

Revenue = (# sold)·(\$12000 − price reduction)

Costs = \$1000 + (# sold)(\$10000/car)

Understand the dynamic:

price reduction = \$200 ⟺ sell 2 more cars

If reduce once, sell 4+2 when price is 12000 − 200.
twice, sell 4+2(2) when price is 12000 − 2(200)
three times, sell 4+2(3) when price is 12000 − 3(200)

If x = # times they reduce the price \$200, then

price = 12000 − 200x

sold = 4 + 2x ← for each decrease

4: original sales; 2x: gain 2 sales

Revenue = (# sold)(price per item) = (4+2x)(12000−200x)

Costs = 1000 + (# sold)(10,000 per) = 1000 + (4+2x)(10,000).

Profits = Revenue − Costs

$$P(x) = (4+2x)(12{,}000-200x) - [1000 + (4+2x)(10000)]$$

Algebra to clean up.

$P(x) = (4+2x)(12000-200x) - [1000 + (4+2x)(10000)]$ Foil, dist 10,000

$= 48000 - 800x + 24000x - 400x^2 - [1000 + 40000 + 20000x]$

$= 48000 + 23200x - 400x^2 - [41000 + 20{,}000x]$ dist neg

$= 48000 + 23200x - 400x^2 - 41{,}000 - 20{,}000x$

$= -400x^2 + 3200x + 7000$

$P'(x) = -800x + 3200$

$P'(x) = 0$

$-800x + 3200 = 0$

$800x = 3200$

$x = 4$

x = # price reductions.

$P''(x) = -800$ $\frown$ max ✓

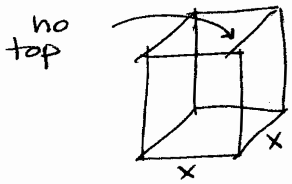
price per car = $12000 - 200x$

price when $x = 4$ $\quad 12000 - 200(4)$

$= \boxed{\$11200}$

(3) An open-top box with a square base is to have a volume of 108 cubic inches. Find the dimensions of the box that can be made with the smallest amount of material.

Objective: Amount of material (min)
$\Rightarrow$ area.
Constraint: Must have volume 108 in^3
Question: dimensions of box
$\Rightarrow$ length, width, height.



base is square

material needed = one square base + 4 rectangular sides.

volume of box must equal 108 in^3.

Variables: x = side of square base
y = height of box.

volume of a rectangular solid = $L \cdot W \cdot H = x \cdot x \cdot y = x^2 y$

$$108 = x^2 y. \quad \leftarrow \text{(constraint)}$$

$$\text{material} = x^2 + 4xy \quad \leftarrow \text{(objective)}$$

$\uparrow$
too many variables.
Use constraint to substitute for y.

Solve $108 = x^2 y$ for y

$$\frac{108}{x^2} = \frac{x^2 y}{x^2}$$

$$108x^{-2} = y$$

Substitute into objective

$$M(x) = x^2 + 4x \cdot 108x^{-2}$$

mult coefficients, add exp.

$$M(x) = x^2 + 432x^{-1}$$

$$M'(x) = 2x - 432x^{-2}$$

$$M'(x) = 0$$

$$2x - \frac{432}{x^2} = 0$$

$$2x = \frac{432}{x^2}$$

$$2x^3 = 432$$

$$x^3 = 216$$

$$x = \sqrt[3]{216}$$

$$x = 6$$

$$M''(x) = 2 + 864x^{-3}$$

$$M''(6) = 2 + 864(6)^{-3} = 6 > 0 \quad \cup \text{ min.}$$

$x = 6$ side of base

$y = \frac{108}{x^2}$ subst $x = 6$

$$y = \frac{108}{6^2}$$

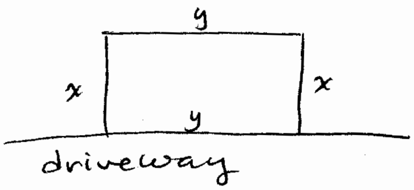
$$y = 3$$

dimensions $6'' \times 6'' \times 3''$

(4) A homeowner builds a rectangular fenced garden with one edge along the driveway. The garden is 5000 ft². The fencing along the driveway costs \$6 per foot. The other 3 sides have fencing costing \$2/foot. Find the dimensions that minimize cost and the minimum cost.

Objective: cost (min)

Question: dimensions of rectangle



y = length along driveway

x = other side length.

Cost = cost of fence by driveway + cost of 1st side + Cost 2nd side + cost 3rd side

cost each side = (length of side in feet) × (cost per foot of fence)

$$\text{Cost} = (\$6)\cdot y + 2x + 2y + 2x \quad \leftarrow \text{(objective)}$$

$$= 6y + 2x + 2y + 2x$$

$$= 8y + 4x$$

⇑

Too many variables.

$$\text{Area} = \text{Length} \times \text{Width} = x \cdot y = 5000 \quad \leftarrow \text{(constraint)}$$

Solve constraint for y: $y = \dfrac{5000}{x}$

$$y = 5000x^{-1}$$

Subst into objective

$$C(x) = 8(5000x^{-1}) + 4x$$

$$C(x) = 40000x^{-1} + 4x$$

$$C'(x) = -40000x^{-2} + 4$$

$$C'(x) = 0$$

$$-\frac{40000}{x^2} + 4 = 0$$

$$4 = \frac{40000}{x^2}$$

$4x^2 = 40000$

$x^2 = 10000$

$x = \pm 100$

but negative length makes no sense!

$c''(x) = 80000x^{-3}$

$c''(100) = \frac{80\,000}{100^3} = .08 > 0$ $\cup$ min.

$y = \frac{5000}{x}$

$y = \frac{5000}{100}$

$y = 50$

dimensions $\boxed{\text{50 ft along driveway} \times \text{100 ft other side/direction}}$